Fractal Methods in Computer Graphics

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Fractal as attractor

Let (X, d) be a complete metric space. Moreover, let $\mathcal{H}(X)$ denote the space of non-empty, compact subsets of X.

Definition

Let (X, d) be a complete metric space, $x \in X$ and $B \in \mathcal{H}(X)$. We extend the metric d to $X \times \mathcal{H}(X)$ in the following way

$$d(x, B) = \min\{d(x, y) : y \in B\}.$$

We say that $d(x, B)$ is the distance of a point x to the set B.

Definition

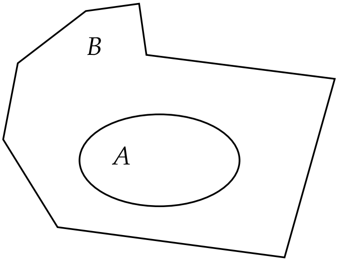
Let (X, d) be a complete metric space and let $A, B \in \mathcal{H}(X)$. We extend the metric d to $\mathcal{H}(X) \times \mathcal{H}(X)$ in the following way

$$d(A, B) = \max\{d(x, B) : x \in A\}.$$

We say that $d(A, B)$ is the distance from A to B.

Let us note that such an extension of d is not a metric in $\mathcal{H}(X)$ yet. It does not fulfil the symmetry condition.

Let us consider a metric space $(\mathbb{R}^2, d_e)$ and any two non-empty, compact sets $A, B \in \mathcal{H}(\mathbb{R}^2)$ such that $A \subsetneq B$, e.g.,



Because $A \subsetneq B$, so for all $x \in A$ we have $d(x, B) = \min\{d_e(x, y) : y \in B\} = 0$. Thus, $d(A, B) = 0$. On the other hand, there exist $y \in B$ such that for all $x \in A$ we have $d_e(y, x) > 0$. From this, we get that $d(y, A) > 0$, and in consequence $d(B, A) > 0$. Therefore, $d(A, B) \neq d(B, A)$.

Definition

Let (X, d) be a complete metric space. Let us define a function $h_X : \mathcal{H}(X) \times \mathcal{H}(X) \to [0, +\infty)$ in the following way

$$\forall_{A,B \in \mathcal{H}(X)} \quad h_X(A, B) = \max\{d(A, B), d(B, A)\}.$$

The function h_X is called the Hausdorff metric.

Theorem
Let (X, d) *be a complete metric space. Then, the space* $(\mathcal{H}(X), h)$ *is a complete metric space.*

Lemma
Let (X, d) *be a metric space and let* $w : X \to X$ *be a contraction mapping with the contractivity factor* s*. Then, the extension* $w : \mathcal{H}(X) \to \mathcal{H}(X)$ *defined by*

$$\forall_{B \in \mathcal{H}(X)} \quad w(B) = \{w(x) : x \in B\}$$

is a contraction mapping on $(\mathcal{H}(X), h)$ *with contractivity factor* s*.*

Lemma

Let (X, d) *be a metric space and let* $\{w_n : n = 1, 2, \ldots, N\}$ *be contraction mappings on* $(\mathcal{H}(X), h)$*. Denote by* s_n *the contractivity factor of* w_n *for* $n = 1, 2, \ldots, N$*. Let us define* $W : \mathcal{H}(X) \to \mathcal{H}(X)$ *in the following way*

$$\forall_{B \in \mathcal{H}(X)} \quad W(B) = w_1(B) \cup w_2(B) \cup \ldots \cup w_N(B) = \bigcup_{n=1}^{N} w_n(B).$$

Then, W *is a contraction mapping with contractivity factor* $s = \max\{s_n : n = 1, 2, \ldots, N\}$*.*

The mapping W from the lemma is called the Hutchinson operator.

Definition

An Iterated Function System (IFS) consists of a complete metric space (X, d) and a finite set of contraction mappings $w_n : X \to X$ with contractivity factors s_n for $n = 1, \ldots, N$. We will denote it in the form $\{X; w_n, n = 1, \ldots, N\}$, and its contractivity factor as $s = \max\{s_n : n = 1, \ldots, N\}$.

Theorem

Let $\{X; w_n, n = 1, \ldots, N\}$ be an IFS with contractivity factor s. Then, the mapping $W : \mathcal{H}(X) \to \mathcal{H}(X)$ defined by

$$W(B) = \bigcup_{n=1}^{N} w_n(B)$$

for all $B \in \mathcal{H}(X)$ is a contraction mapping on a complete metric space $(\mathcal{H}(X), h)$ with contractivity factor s, i.e.,

$$h(W(B), W(C)) \leq s \cdot h(B, C)$$

for all $B, C \in \mathcal{H}(X)$.
Moreover, mapping W has a unique fixed point $A \in \mathcal{H}(X)$, i.e., $A = W(A)$, and it is given by the formula $A = \lim_{k\to\infty} B_k$, where $B_k = W^k(B)$ for any $B \in \mathcal{H}(X)$.

The notation W^k in the theorem means the k-times composition of W.

Definition

The fixed point $A \in \mathcal{H}(X)$ in the theorem from the previous slide is called an attractor of the IFS or the fractal.

So far, the considerations had a general character, so in practical applications of fractals we need to take a specific metric space and a concrete class of contraction mappings.

Generally, in practice we use the $(\mathbb{R}^n, d_e)$ space, and for the contraction mappings we use the affine mappings $w : \mathbb{R}^n \to \mathbb{R}^n$ of the form:

$$w(x) = Ax + t,$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^n_n$, $t \in \mathbb{R}^n$.

Of course, not all affine mappings are contractive, so we need some condition that will guarantee the contractivity.

Let us take and $x, y \in \mathbb{R}^n$. Then

$$\|w(x) - w(y)\| = \|Ax - Ay\| = \|A(x - y)\|.$$

Because mapping of the form Ax is linear and continuous, so

$$\|w(x) - w(y)\| \leq \|A\|\|x - y\|.$$

Thus, it is sufficient that $\|A\| < 1$ for w to be a contraction mapping. If we take the following matrix norm

$$\|A\| = \sqrt{\lambda_{\max}(A^T A)},$$

where $\lambda_{\max}(A^T A)$ is the largest eigenvalue of $A^T A$, then the condition takes the following form:

$$\sqrt{\lambda_{\max}(A^T A)} < 1.$$

Let us recall what the eigenvalue of a matrix is.

Definition

A scalar $\lambda \in \mathbb{R}$ is called eigenvalue of a matrix $A \in \mathbb{R}_n^n$ if there exists a non-zero vector $v \in \mathbb{R}^n$ such that

$$Av = \lambda v.$$

Vector v is called the eigenvector that corresponds to the eigenvalue λ.
The set of all eigenvalues of A is called the spectrum.

Generation of an attractor given by IFS

In the literature we can find various algorithms for the generation of an attractor of a given IFS, e.g.,

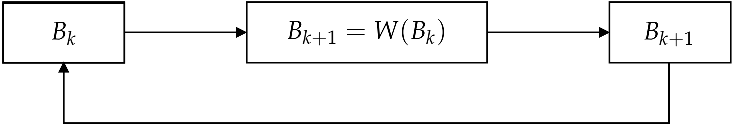
- deterministic method,
- random method,
- minimal plotting method,
- escape time method,
- image based method,
- etc.

A survey of the various methods can be found in the book: Nikiel, S.: Iterated Function Systems for Real-time Image Synthesis. Springer, London, (2007)

Deterministic method

The deterministic method results directly from the definition of the attractor.

Let us assume that we have an IFS $\{X; w_n, n = 1, \ldots, N\}$. We select any starting set $B \in \mathcal{H}(X)$. Next, we perform iterations $B_k = W^k(B)$ for $k = 0, 1, \ldots$.



The more iterations we perform the better approximation of the attractor we get.

The key role in the convergence of the iteration process plays the contractivity factor of the IFS. The closer to 1 the slower the convergence, and in consequence we need to perform more iterations to get a good approximation of the attractor.

In order to generate an image of the attractor using the deterministic method, we perform several iterations obtaining some (good) approximation of the attractor. After some iteration (other for different IFSs), if we perform further iterations, we will not see any further details of the attractor. This is due to the fact that we use a discrete image with a finite resolution.

Despite its simplicity, the deterministic method is not used in practice very often because the speed of generation is slower than the speed of other methods.

This is a consequence of the fact that with each iteration the number of subsets grows N times, so for instance, in the $k-1$ iteration we have N^{k-1} subsets which are the input for the kth iteration in which after transforming the N^{k-1} subsets we get N^k subsets.

We see that with each iteration the memory requirements for storing the subsets and the number of computations grown exponentially.

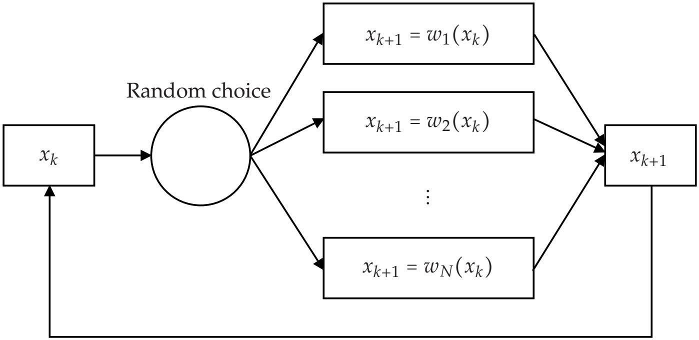
Random method

To introduce the random method (also called the chaos game), we need to define an additional notion.

Definition

An Iterated Function System with probablities consists of an IFS $\{X; w_n, n = 1, 2, \ldots, N\}$ together with a set of probabilities $\{p_1, p_2, \ldots, p_N\}$ such that $p_1 + \ldots + p_N = 1$ and $p_n > 0$ for $n = 1, 2, \ldots, N$. The probability p_n is assigned to the mapping w_n for $n = 1, 2, \ldots, N$. We will denote the IFS with probabilities in the following way $\{X; w_1, w_2, \ldots, w_N; p_1, p_2, \ldots, p_N\}$.

In the chaos game, we have an IFS with probabilities $\{X; w_1, w_2, \ldots, w_N; p_1, p_2, \ldots, p_N\}$ and an arbitrary starting point $x_0 \in X$.



In practice, we are not able to perform infinitely many iterations, so additionally in the algorithm we give the maximal number of iteration that we should perform.

In his book, S.Nikiel approximated the number of iterations needed to obtain a good attractor approximation by $8 \cdot H \cdot V$, where H is the horizontal resolution, and V is the vertical resolution of an image that we generate.

The starting point x_0 can be arbitrarily chosen point.

When the point does not belong to the attractor, then several first points generated by the random process also do not belong to the attractor. After these several iterations, all the other points belong to the attractor. In such a situation, we omit several first iterations in the drawing process.

When the starting point belongs to the attractor, then all the points generated by the random process also belong to the attractor, so we can draw all the generated points.

The choice of the starting point, so it belongs to the attractor, can be performed, for example, by choosing a fixed point of one of the mappings that form the IFS (the existence of such fixed point results from the Banach fixed point theorem).

In the chaos game, we assume that we have an IFS with probabilities. But what to do in a situation in which we do not have such IFS?

When the contraction mappings w_k (in $\mathbb{R}^n$) are affine, then we can calculate the probabilities in such a way that we get an uniform convergence to the attractor.

Let A_k be the mapping matrix of the mapping w_k for $k = 1, 2, \ldots, N$.

We can take the same probability for each of the mappings, but in some cases such choice will not give uniform convergence.

A better choice is to take probabilities calculated with the formula:

$$p_k = \frac{|\det A_k|}{\sum_{i=1}^{N} |\det A_i|} \qquad k = 1, \ldots, N.$$

If for some $k \in \{1, \ldots, N\}$ the determinant $\det A_k = 0$, then as p_k we take some small positive number, e.g., 0.001, and from the other probabilities, we subtract numbers such that the condition $p_1 + \ldots + p_N = 1$ is fulfilled.

When all the determinants are equal to zero, then we select the probabilities empirically or we take the same probabilities for all the mappings.

Examples of attractors

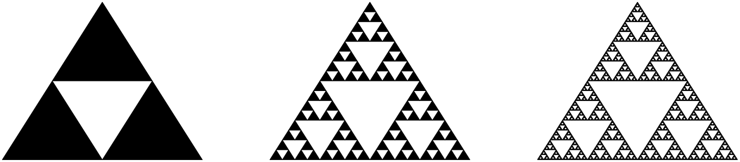
Sierpinski triangle (or gasket) was discovered in 1915. It can be generated by the following IFS $\{\mathbb{R}^2; w_1, w_2, w_3\}$, where

$$w_1(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$w_2(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix},$$

$$w_3(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0.25 \\ 0.5 \end{bmatrix}.$$

Iterations (from left): 1, 4, 8. The starting set: triangle.



Barnsley's fern is given by the following IFS $\{\mathbb{R}^2; w_1, w_2, w_3, w_4\}$:

$$
\begin{aligned}
w_1(\begin{bmatrix} x \\ y \end{bmatrix}) &= \begin{bmatrix} 0.849 & 0.037 \\ -0.037 & 0.849 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0.075 \\ 0.183 \end{bmatrix}, \\
w_2(\begin{bmatrix} x \\ y \end{bmatrix}) &= \begin{bmatrix} 0.197 & -0.226 \\ 0.226 & 0.197 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0.4 \\ 0.049 \end{bmatrix}, \\
w_3(\begin{bmatrix} x \\ y \end{bmatrix}) &= \begin{bmatrix} -0.15 & 0.283 \\ 0.26 & 0.237 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0.575 \\ -0.084 \end{bmatrix}, \\
w_4(\begin{bmatrix} x \\ y \end{bmatrix}) &= \begin{bmatrix} 0 & 0 \\ 0 & 0.16 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}.
\end{aligned}
$$

The probabilities were calculated using the method presented in the previous slides.

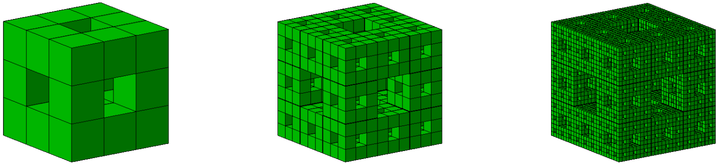
The number of iterations (from left): 1 000, 10 000, 100 000.

The IFS
$\{\mathbb{R}^3; w_{ijk}, (i,j,k) \in \{0,1,2\}^3 \backslash \{(i,j,k) \in \{0,1,2\}^3 : (i = j = 1, k = 0,1,2) \vee (j = k = 1, i = 0,1,2) \vee (i = k = 1, j = 0,1,2)\}\}$
for the Menger sponge is given by the following formula:

$$w_{ijk}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} \frac{i}{3} \\ \frac{j}{3} \\ \frac{k}{3} \end{bmatrix}.$$

The IFS consists of 20 mappings.

Iterations (from left): 1, 2, 3. The starting set: cube.



Inverse fractal problem

So far, we dealt with the problem of how to generate an attractor of a given IFS. Now, we will deal with the inverse problem, i.e., having a given set $A \in \mathcal{H}(X)$ we want to find an IFS for which A is the attractor.

The problem is called the inverse fractal problem. In contrast to the generation of an attractor for which we know many simple and efficient algorithms, the problem of finding an IFS for a given set is a very hard problem.

One of the approaches to this problem was presented by M. Barnsley in his collage theorem.

Theorem (collage theorem)

Let (X, d) be a complete metric space. Let $L \in \mathcal{H}(X)$ and $\varepsilon > 0$ be given. Choose an IFS $\{X; w_1, \dots, w_N\}$ with contractivity factor $0 \leq s < 1$ such that

$$h(L, \bigcup_{n=1}^{N} w_n(L)) \leq \varepsilon.$$

Then

$$h(L, A) \leq \frac{\varepsilon}{1-s},$$

where A is the attractor of the IFS.

Definition

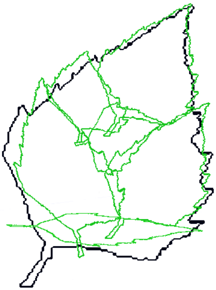
The constant ε from the collage theorem is called the collage error.

The method that results from the collage theorem is the following.

We take the set $L \in \mathcal{H}(X)$ and we cover it with copies of itself obtained by transforming this set by contraction mappings. The coverage is chosen in such a way that the Hausdorff distance between L and the coverage is less or equal to ε.

The IFS obtained in this way has an attractor A that, according to the collage theorem, is distant from L by at most $\frac{\varepsilon}{1-s}$.

In practice, when we use the collage theorem, we require that the number of the contraction mappings that form the IFS is minimal, and that the smaller copies overlap in a minimal way. We make such requirements because in this way we obtain a small number of information needed to generate the attractor.



The following theorem is an important tool among others in computer graphics, and it is closely related to the collage theorem. It determines a continuous dependency between the attractor and the parameters of the mappings that form the IFS.

Theorem

Let (X, d) be a complete metric space. Let $\{X; w_1, \ldots, w_N\}$ be an IFS with contractivity factor s. For $n = 1, \ldots, N$ let w_n depend continuously on the parameter $p \in P$, where (P, d_p) is a compact metric space. Then the attractor $A(p) \in \mathcal{H}(X)$ depends continuously on the parameter $p \in P$ with respect to the Hausdorff metric.

The theorem from the previous slide can be interpreted in the following way: small changes in the parameters of the mappings that form the IFS lead to small changes in the attractor.

It is a very important feature because:

- we can control, in a continuous way, the attractor by changing the parameters of the mappings that form the IFS,
- it guarantees us that we can go from one attractor to another in a continuous way, so we can make the so-called fractal morphing.

Let us take the following IFS $\{\mathbb{R}^2; w_1, w_2, w_3, w_4\}$ which generates the Barnsley's fern, in which the w_1 mapping has one parameter $p \in \mathbb{R}$, where

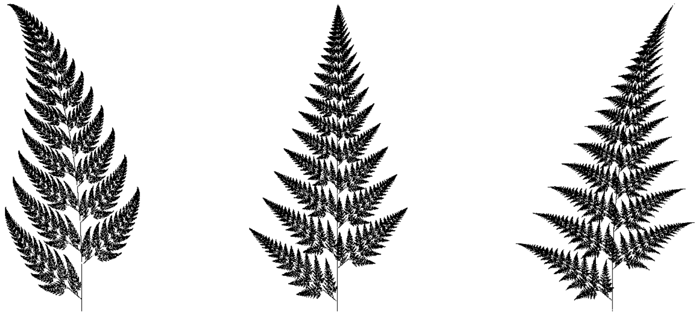
$$w_1(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 0.849 & 0.037 + p \\ -0.037 & 0.849 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0.075 \\ 0.183 \end{bmatrix},$$

$$w_2(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 0.197 & -0.226 \\ 0.226 & 0.197 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0.4 \\ 0.049 \end{bmatrix},$$

$$w_3(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} -0.15 & 0.283 \\ 0.26 & 0.237 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0.575 \\ -0.084 \end{bmatrix},$$

$$w_4(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 0 & 0 \\ 0 & 0.16 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}.$$

Ferns for various values of the p parameter (from left): 0, -0.037, -0.067.



We see a continuous dependency between the attractor and the parameter p of mapping w_1.

In the end, let us notice one important thing. A given attractor can be generated using various IFSs.