

Polynomiography with non-standard iterations

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ABSTRACT

In the paper the visualizations of some modifications applied to the Newton's root finding of complex polynomials are presented. Namely, instead of the standard Picard iteration, several different iterative processes described in the literature, which we call as non-standard ones, were used. Following Kalantari, such visualizations are called polynomiographs. Polynomiographs are interesting from scientific, educational and artistic points of view. By the use of different iterations kinds we obtain quite new polynomiographs, in comparison to the standard Picard iteration, which look aesthetically pleasing. We present some polynomiographs for complex polynomial equation $z^3 - 1 = 0$ as examples. Polynomiographs were defined to graphically present dynamical behaviour of different iterative processes. But we are not interested in that. We are focused on polynomiographs from the artistic point of view. We believe that the new polynomiographs can be interesting as a source of aesthetic patterns created automatically. They can also be used to increase functionality of the existing polynomiography software.

Keywords

polynomiography, iteration process, Newton method, computer art

1 INTRODUCTION

One can meet polynomials in many mathematical fields. They are interesting both from theoretical and practical points of view. Especially, the problem of polynomials root finding has a long and fascinating history. Already Sumerians 3000 years B.C. and ancient Greeks faced with practical problems that in modern mathematical language can be considered as a root finding of polynomials. In 17th century Newton proposed a method for calculating approximately roots of polynomials. Cayley in 1879 observed strange and unpredictable chaotic behaviour of the roots approximation process while applying the Newton's method to the equation $z^3 - 1 = 0$ in the complex plane. The solution of the Cayley's problem was found in 1919 by Julia. Julia sets became an inspiration for the great discoveries in 1970s – the Mandelbrot set and fractals [Man83]. The last interesting contribution to the polynomials root finding history was made by Kalantari [Kal09], who introduced the so-called polynomiography to science. Polynomiography defines

visualization process of the roots of complex polynomials approximation, using fractal and non-fractal images created via the mathematical convergence properties of iteration functions. An individual image is called a polynomiograph. Polynomiography combines both the art and science aspects. Polynomiography, as a method which generates nicely looking graphics, was patented by Kalantari in USA in 2005 [Kal09].

It is known that any complex polynomial p of degree n having n roots, according to the Fundamental Theorem of Algebra, can be uniquely defined by its coefficients $\{a_n, a_{n-1}, \dots, a_1, a_0\}$:

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0 \quad (1)$$

or by its zeros $\{z_1, z_2, \dots, z_{n-1}, z_n\}$:

$$p(z) = (z - z_1)(z - z_2) \dots (z - z_n). \quad (2)$$

Roots finding iterative process can be obviously applied to the both representations of p . As the result the polynomiographs are generated. Degree of polynomial defines the number of basins of attraction (root's basin of attraction is an area of the complex plane in which each point is convergent to the root using the root finding method). Localizations of the basins can be controlled by changing the roots positions on the complex plane manually.

Usually, polynomiographs are coloured based on the number of iterations needed to obtain the approximation of some polynomial root with a given accuracy and

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a chosen iteration method. The description of polynomiography, its theoretical background and artistic applications are described in [Kal05, Kal09].

Fractals and polynomiographs are generated by iterations. Fractals are self-similar, have complicated and non-smooth structure and are not dependent on resolution. Polynomiographs are different. Their shape can be controlled and designed in a more predictable way in opposition to typical fractals. Generally, fractals and polynomiographs belong to different classes of graphical objects.

Summing up, polynomiography can be treated as a visualization tool based on the root finding process. It has many possible applications in education, math, sciences, art and design [Kal09].

In [KGL12] the authors have used Mann and Ishikawa iterations instead of the standard Picard iteration to obtain some generalization of the Kalantari's polynomiography and have presented some polynomiographs for the cubic equation $z^3 - 1 = 0$, permutation and double stochastic matrices. Earlier, other types of iterations were used in [SJM09] for superfractals and in [PK11] for fractals generated by IFS. Also Julia sets and Mandelbrot sets [ARC14] and the so-called antifractals [RC12] were investigated using Noor iteration instead of the standard Picard iteration.

In the paper we generalize the results from [KGL12]. Thanks to the application of the new kinds of iterations we essentially extended the set of polynomiographs. Some of them are very interesting from the aesthetic point of view. They can be used as patterns for textures, in paintings creation, carpet and tapestry design etc.

The paper is organised as follows. In section 2 different types of iterations are defined. Section 3 is devoted to Newton's method for finding roots of polynomials and its generalizations, and presents some iteration formulas. In section 4 examples of polynomiographs with different types of iterations for complex equation $z^3 - 1 = 0$ are presented. The last section, section 5, describes some conclusions and plans for future work.

2 ITERATIONS

It is known that equations of the form $f(x) = 0$ can be equivalently transformed into a fixed point problem $x = T(x)$, where T is some operator [BF11]. Then, applying approximate fixed point theorem one can get information on the existence or sometimes both on existence and uniqueness of fixed point that is the solution of the starting equation.

Let (X, d) be a complete metric space and $T : X \rightarrow X$ a selfmap on X . The set $\{x^* \in X : T(x^*) = x^*\}$ is the set of all fixed points of T . In the ample literature [Ber07, Ish74, KDGE13, Kha13, Noo00, Man53, PS11, Sua05] many iterative processes have been described for the

approximation of fixed points. Below we recall some known iteration processes from the literature. Assume that each iteration process starts from any initial point $x_0 \in X$.

- The standard Picard iteration [Pic90] introduced in 1890 is defined as:

$$x_{n+1} = T(x_n), \quad n = 0, 1, 2, \dots, \quad (3)$$

- The Mann iteration [Man53] was defined in 1953 as:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T(x_n), \quad n = 0, 1, 2, \dots, \quad (4)$$

where $\alpha_n \in (0, 1]$ for all $n \in \mathbb{N}$.

- The Ishikawa iteration [Ish74] was defined in 1974 as a two-step process:

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T(y_n), \\ y_n = (1 - \beta_n)x_n + \beta_n T(x_n), \end{cases} \quad n = 0, 1, 2, \dots, \quad (5)$$

where $\alpha_n \in (0, 1]$ and $\beta_n \in [0, 1]$ for all $n \in \mathbb{N}$.

- The Noor iteration [Noo00] was defined in 2000 as a three-step process as:

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T(y_n), \\ y_n = (1 - \beta_n)x_n + \beta_n T(z_n), \\ z_n = (1 - \gamma_n)x_n + \gamma_n T(x_n), \end{cases} \quad n = 0, 1, 2, \dots, \quad (6)$$

where $\alpha_n \in (0, 1]$ and $\beta_n, \gamma_n \in [0, 1]$ for all $n \in \mathbb{N}$.

- In 2013 Khan iteration [Kha13] was defined as the following process:

$$\begin{cases} x_{n+1} = T(y_n), \\ y_n = (1 - \alpha_n)x_n + \alpha_n T(x_n), \end{cases} \quad n = 0, 1, 2, \dots, \quad (7)$$

where $\alpha_n \in (0, 1]$ for all $n \in \mathbb{N}$.

- SP iteration [PS11] was defined in 2011 as the following three-step process:

$$\begin{cases} x_{n+1} = (1 - \alpha_n)y_n + \alpha_n T(y_n), \\ y_n = (1 - \beta_n)z_n + \beta_n T(z_n), \\ z_n = (1 - \gamma_n)x_n + \gamma_n T(x_n), \end{cases} \quad n = 0, 1, 2, \dots, \quad (8)$$

where $\alpha_n \in (0, 1]$ and $\beta_n, \gamma_n \in [0, 1]$ for all $n \in \mathbb{N}$.

- The Suantai iteration [Sua05] was defined in 2005 as a three-step iteration process with 5 parameters:

$$\begin{cases} x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \alpha_n T(y_n) + \beta_n T(z_n), \\ y_n = (1 - a_n - b_n)x_n + a_n T(z_n) + b_n T(x_n), \\ z_n = (1 - \gamma_n)x_n + \gamma_n T(x_n), \end{cases} \quad n = 0, 1, 2, \dots, \quad (9)$$

where $\alpha_n, \beta_n, \gamma_n, a_n, b_n \in [0, 1]$, $\alpha_n + \beta_n \in [0, 1]$, $a_n + b_n \in [0, 1]$ for all $n \in \mathbb{N}$ and $\sum_{n=0}^{\infty} (\alpha_n + \beta_n) = \infty$.

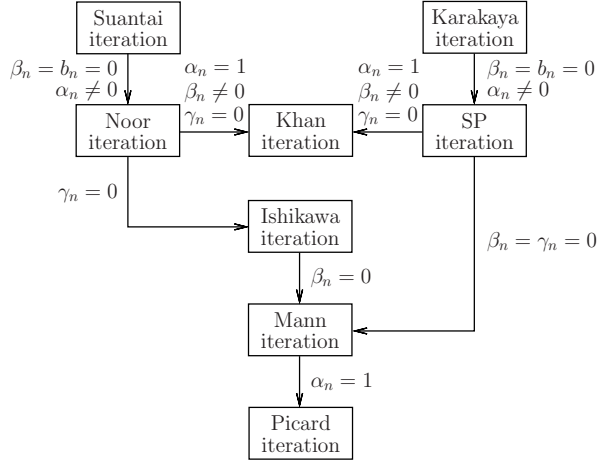


Figure 1: The diagram of iterations dependencies.

- In 2013 Karakaya et al. in [KDGE13] defined very general three-step iteration process with 5 parameters:

$$\begin{cases} x_{n+1} = (1 - \alpha_n - \beta_n)y_n + \alpha_n T(y_n) + \beta_n T(z_n), \\ y_n = (1 - a_n - b_n)z_n + a_n T(z_n) + b_n T(x_n), \\ z_n = (1 - \gamma_n)x_n + \gamma_n T(x_n), \quad n = 0, 1, 2, \dots, \end{cases} \quad (10)$$

where $\alpha_n, \beta_n, \gamma_n, a_n, b_n \in [0, 1]$, $\alpha_n + \beta_n \in [0, 1]$, $a_n + b_n \in [0, 1]$ for all $n \in \mathbb{N}$ and $\sum_{n=0}^{\infty} (\alpha_n + \beta_n) = \infty$.

The standard Picard iteration is used in the Banach Fixed Point Theorem [Ber07] to obtain the existence of the fixed point x^* of the operator T . The fixed point approximation is found under additional assumptions on the space X that it should be a Banach one and the mapping T should be contractive. The Mann [Man53], Ishikawa [Ish74] and other iterations [Ber07, KDGE13, Kha13, Noo00, PS11, Sua05] allow to weak the assumptions on the mapping T and generally allow to approximate fixed points. Dependencies between different types of iterations are presented in Fig. 1.

Our further considerations will be conducted in the space $X = \mathbb{C}$ that is obviously a Banach one. We take $z_0 \in \mathbb{C}$ and $\alpha_n = \alpha$, $\beta_n = \beta$, $\gamma_n = \gamma$, $a_n = a$, $b_n = b$ for all $n \in \mathbb{N}$ such that $\alpha \in (0, 1]$, $\beta, \gamma, a, b \in [0, 1]$, $\alpha + \beta \in [0, 1]$ and $a + b \in [0, 1]$.

3 NEWTON ROOT FINDING METHOD AND ITS GENERALIZATIONS

In this section we recall the well-known Newton method for finding roots of a complex polynomial p . The Newton procedure is given by the formula:

$$z_{n+1} = N(z_n), \quad n = 0, 1, 2, \dots, \quad (11)$$

where $N(z) = z - \frac{p(z)}{p'(z)}$, $p'(z)$ is the first derivative of p at z and $z_0 \in \mathbb{C}$ is a starting point.

Applying the Mann iteration (4) in (11) we obtain the following formula:

$$z_{n+1} = (1 - \alpha)z_n + \alpha N(z_n), \quad n = 0, 1, 2, \dots, \quad (12)$$

where $\alpha \in (0, 1]$.

Using the Ishikawa iteration (5) in (11) we get:

$$\begin{cases} z_{n+1} = (1 - \alpha)z_n + \alpha N(v_n), \\ v_n = (1 - \beta)z_n + \beta N(z_n), \quad n = 0, 1, 2, \dots, \end{cases} \quad (13)$$

where $\alpha \in (0, 1]$ and $\beta \in [0, 1]$.

Substituting the Noor iteration (6) in (11) we get:

$$\begin{cases} z_{n+1} = (1 - \alpha)z_n + \alpha N(v_n), \\ v_n = (1 - \beta)z_n + \beta N(w_n), \\ w_n = (1 - \gamma)z_n + \gamma N(z_n), \quad n = 0, 1, 2, \dots, \end{cases} \quad (14)$$

where $\alpha \in (0, 1]$ and $\beta, \gamma \in [0, 1]$.

Using the Khan iteration (7) in (11) we get:

$$\begin{cases} z_{n+1} = N(v_n), \\ v_n = (1 - \alpha)z_n + \alpha N(z_n), \quad n = 0, 1, 2, \dots, \end{cases} \quad (15)$$

where $\alpha \in (0, 1]$.

Substituting SP iteration (8) in (11) we get:

$$\begin{cases} z_{n+1} = (1 - \alpha)v_n + \alpha N(v_n), \\ v_n = (1 - \beta)w_n + \beta N(w_n), \\ w_n = (1 - \gamma)z_n + \gamma N(z_n), \quad n = 0, 1, 2, \dots, \end{cases} \quad (16)$$

where $\alpha \in (0, 1]$ and $\beta, \gamma \in [0, 1]$.

Using the Suantai iteration (9) in (11) we get:

$$\begin{cases} z_{n+1} = (1 - \alpha - \beta)z_n + \alpha N(v_n) + \beta N(w_n), \\ v_n = (1 - a - b)z_n + aN(w_n) + bN(z_n), \\ w_n = (1 - \gamma)z_n + \gamma N(z_n), \quad n = 0, 1, 2, \dots, \end{cases} \quad (17)$$

where $\alpha \in (0, 1]$, $\beta, \gamma, a, b \in [0, 1]$, $\alpha + \beta \in (0, 1]$, $a + b \in (0, 1]$.

Applying the Karakaya iteration (10) in (11) we get:

$$\begin{cases} z_{n+1} = (1 - \alpha - \beta)v_n + \alpha N(v_n) + \beta N(w_n), \\ v_n = (1 - a - b)w_n + aN(w_n) + bN(z_n), \\ w_n = (1 - \gamma)z_n + \gamma N(z_n), \quad n = 0, 1, 2, \dots, \end{cases} \quad (18)$$

where $\alpha \in (0, 1]$, $\beta, \gamma, a, b \in [0, 1]$, $\alpha + \beta \in (0, 1]$, $a + b \in (0, 1]$.

The sequence $\{z_n\}_{n=0}^{\infty}$ (or orbit of the point z_0) either converges or does not to a root of p . If the sequence converges to a root z^* then we say that z_0 is attracted to z^* . A set of all starting points z_0 for which $\{z_n\}_{n=0}^{\infty}$ converges to z^* is called the basin of attraction of z^* .

Boundaries between basins usually are fractals in nature. In [SK09] some generalizations of the classic Newton formula (11) are discussed.

All the above presented iteration processes are convergent to roots of polynomial p . Only the speed and the character of convergency is different and the basins of attraction to roots of p look differently for different kinds of iterations used. The defined iteration processes are used in the next section to obtain polynomiographs.

The application of non-standard iterations perturbs the shape of polynomial basins and makes the polynomiographs look more "fractally". The aim of using more general iterations, instead of the Picard iteration, was not to improve the speed of convergence but to create images that are interesting from aesthetic point of view.

4 EXAMPLES OF POLYNOMIOGRAPHS WITH DIFFERENT ITERATIONS

In this section we present some polynomiographs for complex polynomial equation $z^3 - 1 = 0$ using different iterations. In all examples the colour of each point in the image is determined with the help of Algorithm 1. I_q in the algorithm corresponds to the iteration processes from section 3 with a vector of the parameters $q \in \mathbb{R}^N$, where N is the number of parameters of the iteration. For the Picard iteration we use I instead of I_q .

Algorithm 1: Determination of colour

Input: $z_0 \in \mathbb{C}$ – starting point, k – maximum number of iterations, ε – accuracy, $q \in \mathbb{R}^N$ – parameters of the iteration I_q , $colours[0..k]$ – colourmap

Output: colour c of z_0

```

1  $i = 0$ 
2 while  $i \leq k$  do
3    $z_{i+1} = I_q(z_i)$ 
4   if  $|z_{i+1} - z_i| < \varepsilon$  then
5     break
6    $i = i + 1$ 
7  $c = colours[i]$ 

```

In the algorithm for a given point z_0 we iterate that point using I_q iteration process. If the modulus of the difference between two successive points in the iteration process is smaller than the given accuracy $\varepsilon > 0$ we assume that the generated sequence converge to a root of p and we stop the iteration. If we reach the maximum number of iterations k we assume that the generated sequence does not converge to any root of p . At the end we give a colour to the considered point using the iteration number at which we have left the while loop. This type of colouring is called the iteration colouring.

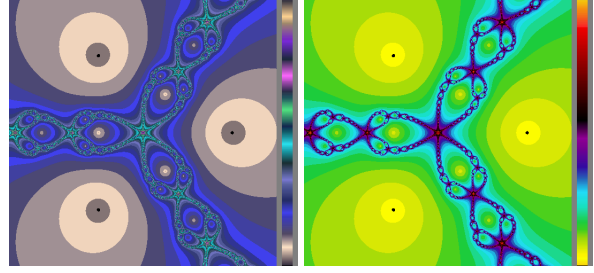


Figure 2: Picard iteration in Newton's method for $z^3 - 1$.

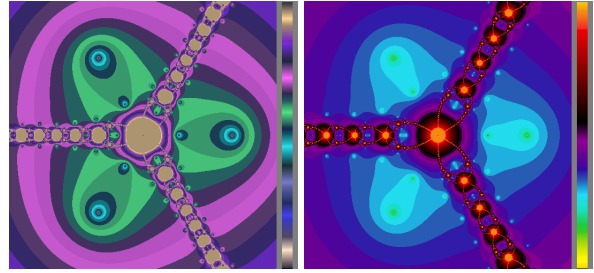


Figure 3: Mann iteration in Newton's method for $z^3 - 1$.

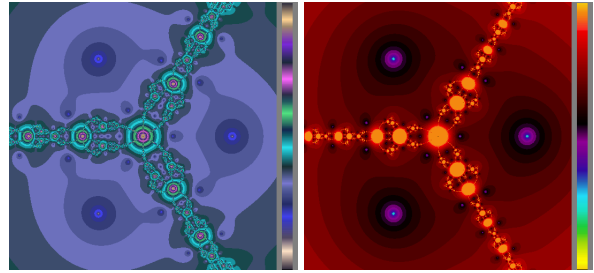


Figure 4: Ishikawa iteration in Newton's method for $z^3 - 1$.

The algorithm with different iterations from section 3 was implemented in the Processing language. Every polynomiograph in this section was computed for 1-5 seconds in average on a laptop with the Intel Core2 Duo 2 GHz CPU, 4 GB RAM.

The cubic equation $z^3 - 1 = 0$ was solved in the square domain $[-1.5, 1.5] \times [-1.5, 1.5]$ using eight different iteration processes from section 3. Images are of resolution 500×500 pixels and were generated with $k = 30$ and $\varepsilon = 0.05$ for two colourmaps. The gradient colour bar has been added to the images that shows how many iterations are needed to obtain the required accuracy ε . The performed experiments showed that values $k = 30$ and $\varepsilon = 0.05$ ensured the acceptable visual quality of polynomiographs. The obtained polynomiographs are presented in Figs. 2–9.

Generally, for polynomiographs in Figs. 2–9 one can observe that for iterations with more parameters images are more complex and are more "fractal". Polynomiographs are strongly dependent not only on iterations but also on the colourmaps used, as it can be easily seen in Fig. 10. The same graphical information con-

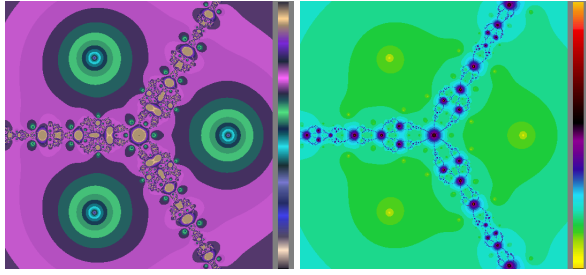


Figure 5: Noor iteration in Newton's method for $z^3 - 1$.

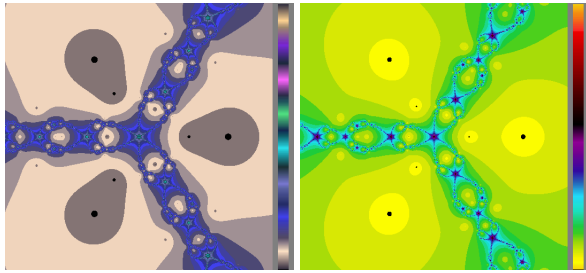


Figure 6: Khan iteration in Newton's method for $z^3 - 1$.

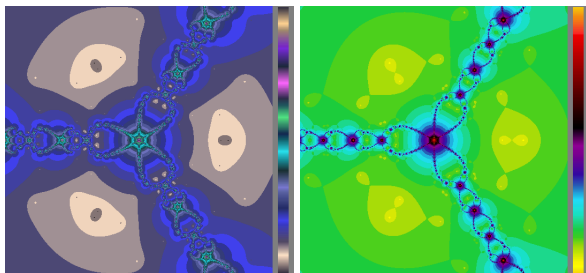


Figure 7: SP iteration in Newton's method for $z^3 - 1$.

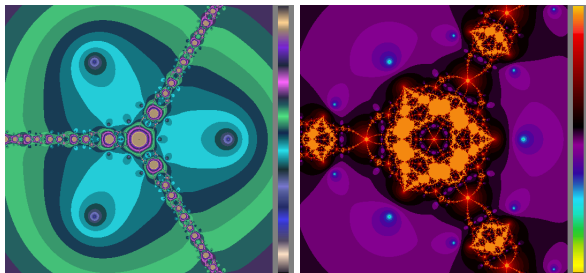


Figure 8: Suantai iteration in Newton's method for $z^3 - 1$.

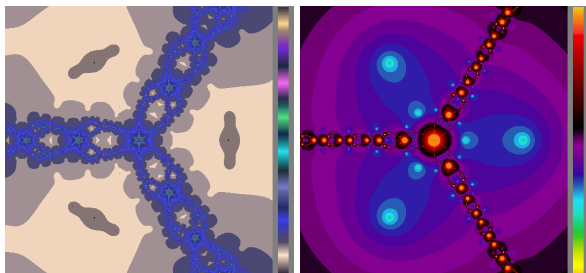


Figure 9: Karakaya iteration in Newton's method for $z^3 - 1$.

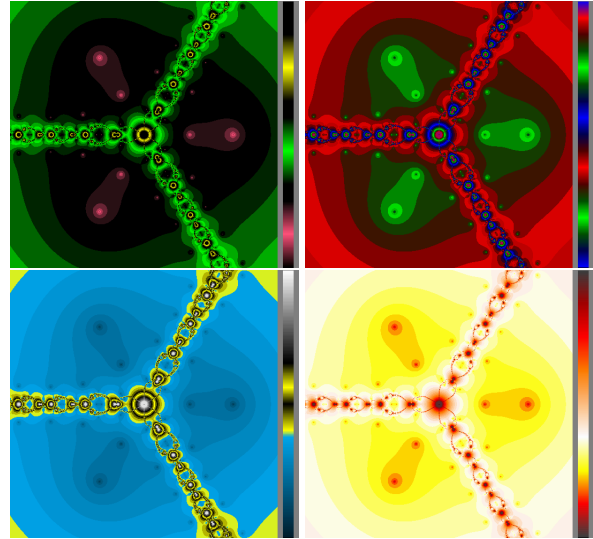


Figure 10: Fixed polynomiograph with different colourmaps.

tained in a polynomiograph may be drastically different for different colourmaps.

It should be stressed that detailed analysis of polynomiographs with respect to parameters of iterations used is difficult. But for practical use it is enough to know that if parameters are constrained and generally lie in the interval $[0, 1]$ then the iteration process is convergent and lead to images that are potentially interesting from the aesthetic point of view.

5 CONCLUSIONS AND FUTURE WORK

In the paper we presented some generalizations of the classic Newton roots finding method with non-standard iterations and some corresponding polynomiographs for exemplary complex polynomial equation $z^3 - 1 = 0$. By changing parameters of different iteration processes one can obtain a huge collection of polynomiographs essentially richer in comparison to the case of the standard Picard iteration. Further generalizations can be obtained with the help of higher order methods based on the Basic and the Euler-Schröder families of iterations [Kal09, Kal11] with non-standard iteration processes. The use of complex valued parameters instead of the real ones, as was checked by the authors, and the use of various convergence tests, as in [Gda13], lead to nice modifications of polynomiographs, as well. Additionally, different colourmaps have great influence on the aesthetic appearance of the polynomiographs.

The above mentioned problems determine our further investigations. We believe that the results of the paper can be interesting to those whose work or hobbies are related to automatically created nicely looking images. In our opinion non-standard iterations can be applied to

increase functionality of the existing polynomiography software, as well.

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