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ORIGINAL PAPER

Convergence analysis of Picard–SP iteration process for generalized α–nonexpansive mappings

Bashir Nawaz[1] · **Kifayat Ullah**[1] · **Krzysztof Gdawiec**[2]

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Abstract
In this manuscript, we introduce a novel hybrid iteration process called the Picard–SP iteration process. We apply this new iteration process to approximate fixed points of generalized α–nonexpansive mappings. Convergence analysis of our newly proposed iteration process is discussed in the setting of uniformly convex Banach spaces and results are correlated with some other existing iteration processes. The dominance of the newly proposed iteration process is exhibited with the help of a new numerical example. In the end, the comparison of polynomiographs generated by other well-known iteration processes with our proposed iteration process has been presented to make a strong impression of our proposed iteration process.

Keywords Generalized α-nonexpansive mapping · Fixed point · Iteration scheme · Polynomiography

1 Introduction and preliminaries

Banach [4] outlined a very basic idea of contraction mapping and proved the well-known Banach contraction principle (BCP). This result is the basis of fixed point theory, which guarantees not only the fixed point of contraction mapping but also the uniqueness of the fixed point. Browder [7], Gohde [13], and Kirk [19] extended the

✉ Krzysztof Gdawiec
krzysztof.gdawiec@us.edu.pl

Bashir Nawaz
bashir.pm@ulm.edu.pk

Kifayat Ullah
kifayatmath@yahoo.com

1 Department of Mathematics, University of Lakki Marwat, Lakki Marwat, 28420 Khyber Pakhtunkhwa, Pakistan

2 Institute of Computer Science, University of Silesia in Katowice, Bedzinska 39, 41-200 Sosnowiec, Poland

idea of Banach and introduced new research dimensions in the field of fixed point theory.

Definition 1 Let U be a nonempty subset of a Banach space X. A mapping $S : U \to U$ is called contraction if for all $p, \mathfrak{z} \in U$, there exists $\vartheta \in [0, 1)$ such that:

$$||Sp - S\mathfrak{z}|| \leq \vartheta ||p - \mathfrak{z}||. \tag{1}$$

For $\vartheta = 1$, (1) is termed as nonexpansive mapping. The point satisfying $St = t$ for any arbitrary $t \in U$ is known as a fixed point of mapping S. In this paper, $Fix(S)$ will represent the set of all fixed points of the mapping S.

With the passage of time, efforts have been made to introduce mappings weaker than contraction mapping. Zamfirescu [37] introduced Zamfirescu mappings that serve as an important generalization for BCP. In [5], Berinde gave a more general class of mappings known as quasi-contractive mappings. Following this, Imoru and Olantiwo [15] gave the following definition.

Definition 2 An operator S is called a contractive-like operator if for each $p, \mathfrak{z} \in U$, there exists a constant $\vartheta \in [0, 1)$ and strictly increasing and continuous function $\xi : [0, \infty) \to [0, \infty)$ with $\xi(0) = 0$ such that

$$\|Sp - S\mathfrak{z}\| \leq \xi(\|p - Sp\|) + \vartheta \|p - \mathfrak{z}\|. \tag{2}$$

In [33], Suzuki introduced a new class of maps with weaker condition than nonexpnasive maps and named that as Condition (C).

Definition 3 A mapping $S : U \to U$ is said to be a mapping satisfying Condition (C) if

$$\frac{1}{2}\|p - Sp\| \leq \|p - \mathfrak{z}\| \Rightarrow \|Sp - S\mathfrak{z}\| \leq \|p - \mathfrak{z}\|, \ \forall p, \mathfrak{z} \in U. \tag{3}$$

In [3], Ayoama and Kahsoka suggested another generalization of contraction, that is α–nonexpansive mapping.

Definition 4 A mapping $S : U \to U$ is called α–nonexpansive mapping if for each $p, \mathfrak{z} \in U$, there exists $\alpha < 1$ such that

$$||Sp - S\mathfrak{z}||^2 \leq \alpha||p - S\mathfrak{z}||^2 + \alpha||\mathfrak{z} - Sp||^2 + (1 - 2\alpha)||p - \mathfrak{z}||^2. \tag{4}$$

In [26], Pant and Shukla proposed the notion of generalized α–nonexpansive mapping.

Definition 5 Mapping $S : U \to U$ is called generalized α–nonexpansive mapping if for each $p, \mathfrak{z} \in U$, there exist $\alpha \in [0, 1)$ such that

$$\frac{1}{2}||p-Sp|| \leq ||p-\mathfrak{z}|| \Rightarrow ||Sp-S\mathfrak{z}|| \leq \alpha||p-S\mathfrak{z}||+\alpha||\mathfrak{z}-Sp||+(1-2\alpha)||p-\mathfrak{z}||. \tag{5}$$

Proposition 1 [26] *Let U be a closed and nonempty subset of a Banach space X, then following results hold for any selfmap S on U:*

(i) If S satisfies Condition (C), then S is generalized α–nonexpansive, but the converse is not true.
(ii) If $Fix(S) \neq \emptyset$ and S is generalized α–nonexpansive, then

$$||Sp - t|| \leq ||p - t||, \text{ for } p \in U, t \in Fix(S).$$

(iii) If the Banach space X is strictly convex, $U \subseteq X$ is convex and S is generalized α–nonexpansive, then $Fix(S)$ is closed and convex.
(iv) If S is generalized α–nonexpansive, then

$$||p - S\mathfrak{z}|| \leq \left(\frac{3+\alpha}{1-\alpha}\right)||p - Sp|| + ||p - \mathfrak{z}||, \; \forall p, \mathfrak{z} \in U.$$

(v) Let $U \subseteq X$ be equipped with Opial's property and S is generalized α–nonexpansive mapping. If $\{m_i\}$ converges weakly to t and $\lim_{i\to\infty} \|Sm_i - m_i\| = 0$, then $St = t$.

Definition 6 ([12]) A space X is termed as uniformly convex Banach space (UCBS) if for each $\varsigma \in (0, 2]$ there exist $\Im > 0$ such that for $p, \mathfrak{z} \in X$,

$$\left.\begin{array}{r} \|p\| \leq 1 \\ \|\mathfrak{z}\| \leq 1 \\ \|p - \mathfrak{z}\| > \varsigma \end{array}\right\} \Rightarrow \left\|\frac{p-\mathfrak{z}}{2}\right\| \leq \Im. \tag{6}$$

Definition 7 ([12]) The modulus of convexity of a Banach space X is the function $\varsigma_X : [0, 2] \to [0, 1]$ defined by

$$\varsigma_X(\Im) = \inf\left\{1 - \left\|\frac{p+\mathfrak{z}}{2}\right\| : \|p\| \leq 1, \; \|\mathfrak{z}\| \leq 1, \; \|p - \mathfrak{z}\| \geq \Im\right\}.$$

Definition 8 Let $\{m_i\}$ be a bounded sequence in a Banach space X. If $\emptyset \neq U \subseteq X$ is convex and closed, then the asymptotic radius of $\{m_i\}$ corresponding to U can be described as

$$r(U, \{m_i\}) = \inf\{\limsup_{i\to\infty} ||m_i - p|| : p \in U\}.$$

Similarly, the asymptotic center of the sequence $\{m_i\}$ corresponding to U is explained by the formula

$$\mathscr{A}(U, \{m_i\}) = \{p \in U : \limsup_{i\to\infty} ||m_i - p|| = r(U, \{m_i\})\}.$$

Remark 1 [8] If X denotes a UCBS, then it is well-known that $\mathscr{A}(U, \{m_i\})$ contains only one element. Also note that when U is convex as well as weakly compact then $\mathscr{A}(U, \{m_i\})$ is convex space (see, e.g. [29, 34] and others).

Definition 9 [24] Let X be a Banach space. Then, for every sequence $\{m_i\}$ in X that converges weakly to $\mathfrak{z} \in X$, the following inequality is satisfied:

$$\lim_{i\to\infty} \sup ||m_i - \mathfrak{z}|| < \lim_{i\to\infty} \sup ||m_i - \tilde{y}||, \ \forall \, \tilde{y} \in P, \ \text{where } \mathfrak{z} \neq \tilde{y}.$$

Definition 10 [32] A self mapping S defined on a subset $U \subseteq X$ is equipped with Condition (I) if there exists a function $g : [0, \infty) \to [0, \infty)$ such that g is non-decreasing with $g(0) = 0$, $g(x) > 0$ for every $x > 0$ and $||q - Sq|| \geq g(d(q, Fix(S)))$, for each $q \in U$, where $d(q, Fix(S)) = \inf_{t \in Fix(S)} \|q - t\|$.

Definition 11 [6] Let $\{\mathfrak{g}_i\}$ and $\{\mathfrak{f}_i\}$ be real convergent sequences with limits $\mathfrak{g}$ and $\mathfrak{f}$, respectively. If $\lim\limits_{i\to\infty} |\frac{\mathfrak{g}_i - \mathfrak{g}}{\mathfrak{f}_i - \mathfrak{f}}| = 0$, then $\{\mathfrak{g}_i\}$ is said to converge faster than $\{\mathfrak{f}_i\}$.

Lemma 1 [31] *Let* $\{\mathfrak{m}_i\}$ *be any real sequence such that* $0 < j \leq \mathfrak{m}_i \leq k < 1$, *for every choice of* $i \geq 1$. *If* $\{\mathfrak{x}_i\}$ *and* $\{y_i\}$ *are any two sequences in a UCBS* X *with* $\lim\limits_{i\to\infty} \sup \|\mathfrak{x}_i\| \leq r$ *and* $\lim\limits_{i\to\infty} \sup \|y_i\| \leq r$, $\lim\limits_{i\to\infty} \sup \|(1 - \mathfrak{m}_i)\mathfrak{x}_i + \mathfrak{m}_i y_i\| = r$ *for a real number* $r \geq 0$, *then* $\lim\limits_{i\to\infty} \|\mathfrak{x}_i - y_i\| = 0$.

Numerical computation of nonlinear operator is very famous and important research area for modern day researcher. Fixed point approximation of nonlinear operators required pre-eminent approach. Initially, Picard [28] iteration process was used along with BCP for fixed point approximation of mappings satisfying contraction condition but in general Picard iteration process is not effective in the case of nonexpansive mappings. To cope this issue, different single step and two step iteration processes, see for example [1, 2, 16, 22], have been introduced in the literature for fixed point estimation of nonexpansive (also generalized nonexpansive) mappings.

Picard [28], gave the idea of the Picard iteration process, which generates a sequence $\{m_i\}$ for initial value $m_0 \in U$, defined as

$$m_{i+1} = Sm_i. \tag{7}$$

Let $\{\psi_i\}$, $\{\mu_i\}$, $\{\xi_i\}$ denote real sequences in $(0, 1]$. In [23], Noor introduced first three-step iteration process, the Noor iteration process, which generates the sequence $\{m_i\}$ given as:

$$\begin{cases} m_0 \in U, \\ w_i = (1 - \xi_i)m_i + \xi_i Sm_i, \\ v_i = (1 - \mu_i)m_i + \mu_i Sw_i, \\ m_{i+1} = (1 - \psi_i)m_i + \psi_i Sv_i. \end{cases} \tag{8}$$

In 2011, Phuentgrattana and Sunatai (see [27]) introduced three step iteration process, the SP iteration process, which generates the sequence $\{m_i\}$ defined as:

$$\begin{cases} m_0 \in U, \\ w_i = (1-\xi_i)m_i + \xi_i S m_i, \\ v_i = (1-\mu_i)w_i + \mu_i S w_i, \\ m_{i+1} = (1-\psi_i)v_i + \psi_i S v_i. \end{cases} \tag{9}$$

In 2011, Sunatai (see [30]) introduced a three-step iteration process, denoted as P iteration process, which generates the sequence $\{m_i\}$ as follows:

$$\begin{cases} m_0 \in U, \\ w_i = (1-\xi_i)m_i + \xi_i S m_i, \\ v_i = (1-\mu_i)w_i + \mu_i S w_i, \\ m_{i+1} = (1-\psi_i)S m_i + \psi_i S v_i. \end{cases} \tag{10}$$

In 2018, Daengsaen and Khempet (see [9]) suggested new three-step iteration process called D iteration process, which generates the sequence $\{m_i\}$ given as:

$$\begin{cases} m_0 \in U, \\ w_i = (1-\xi_i)m_i + \xi_i S m_i, \\ v_i = (1-\mu_i)S m_i + \mu_i S w_i, \\ m_{i+1} = (1-\psi_i)S w_i + \psi_i S v_i. \end{cases} \tag{11}$$

In 2019, Kanayo Stella and Husdson (see [10]) gave the idea of four-step iteration process called Picard–Noor iteration process, which generates the sequence $\{m_i\}$ given as:

$$\begin{cases} m_0 \in U, \\ w_i = (1-\xi_i)m_i + \xi_i S m_i, \\ v_i = (1-\mu_i)m_i + \mu_i S w_i, \\ l_i = (1-\psi_i)m_i + \psi_i S v_i, \\ m_{i+1} = S l_i. \end{cases} \tag{12}$$

In 2021, Lamba and Panwar (see [21]) suggested four-step iteration process called Picard–S* iteration process, which generates the sequence $\{m_i\}$ defined as:

$$\begin{cases} m_0 \in U, \\ w_i = (1-\xi_i)m_i + \xi_i S m_i, \\ v_i = (1-\mu_i)S m_i + \mu_i S w_i, \\ l_i = (1-\psi_i)S m_i + \psi_i S v_i, \\ m_{i+1} = S l_i. \end{cases} \tag{13}$$

2 Main results

Gradually improvements in iterations is based on better convergence results. We have observed that hybrid models of any iteration process significantly generate improved convergent result as compared to that iteration process. This versatility of fixed point theory field has strongly inspired us to introduce new hybrid iteration process. We have proposed a new iteration process called Picard–SP iteration process, which generates the sequence $\{m_i\}$ defined as:

$$\begin{cases} m_0 \in U, \\ a_i = (1 - \xi_i)m_i + \xi_i S m_i, \\ b_i = (1 - \mu_i)a_i + \mu_i S a_i, \\ c_i = (1 - \psi_i)b_i + \psi_i S b_i, \\ m_{i+1} = S c_i. \end{cases} \tag{14}$$

where $\{\psi_i\}$, $\{\mu_i\}$, $\{\xi_i\}$ are sequences in (0, 1] for $i \in \mathbb{N}$.

We first prove a fundamental lemma using our new iteration process before establishing the main results.

Lemma 2 *Let X be any UCBS and $\emptyset \neq U \subseteq X$ be closed and convex. If a selfmap $S : U \to U$ is generalized α–nonexpansive mapping with $Fix(S) \neq \emptyset$ and $\{m_i\}$ is a sequence generated by the Picard–SP iteration process (14), then for all $t \in Fix(S)$, the limit $\lim_{i\to\infty} ||m_i - t||$ exists.*

Proof We may choose any $t \in Fix(S)$. Using (14) with Proposition 1 (ii), we get

$$\begin{aligned} ||a_i - t|| &= ||(1 - \xi_i)m_i + \xi_i S m_i - t|| \\ &\leq (1 - \xi_i)||m_i - t|| + \xi_i ||S m_i - t|| \\ &\leq (1 - \xi_i)||m_i - t|| + \xi_i ||m_i - t|| \\ &= ||m_i - t||. \end{aligned} \tag{15}$$

Similarly, we can prove that

$$||b_i - t|| \leq ||a_i - t||, \tag{16}$$

and,

$$||c_i - t|| \leq ||b_i - t||. \tag{17}$$

Also,

$$||m_{i+1} - t|| = ||S c_i - t|| \leq ||c_i - t||. \tag{18}$$

Using (15), (16), (17) in (18), we obtain

$$||m_{i+1} - t|| \leq ||c_i - t|| \leq ||b_i - t|| \leq ||a_i - t|| \leq ||m_i - t||. \tag{19}$$

We can conclude that $||m_{i+1} - t|| \leq ||m_i - t||$, i.e $\{||m_i - t||\}$ is nonincreasing and bounded. This implies that $\lim_{i\to\infty} ||m_i - t||$ exists for each $t \in Fix(S)$. □

Now, we explain important condition for fixed point existence of generalized α–nonexpansive mappings.

Theorem 1 *Assume that S is a generalized α–nonexpansive mapping defined on a nonempty closed subset U of a UCBS X, and let $\{m_i\}$ be a sequence generated by* (14)*, then*

$$Fix(S) \neq \emptyset \iff \{m_i\} \text{ is bounded and } \lim_{i\to\infty} ||Sm_i - m_i|| = 0. \tag{20}$$

Proof Assume that $Fix(S)$ is not empty, and let $t \in Fix(S)$.

Lemma 2 assures that $\lim_{i\to\infty} ||m_i - t||$ exists as well as $\{m_i\}$ is bounded. Now, we will prove that $\lim_{i\to\infty} ||Sm_i - m_i|| = 0$. For this, suppose that

$$\lim_{i\to\infty} ||m_i - t|| = \varepsilon. \tag{21}$$

From Lemma 2, we have

$$\begin{aligned} ||a_i - t|| &\leq ||m_i - t||, \\ \limsup_{i\to\infty} ||a_i - t|| &\leq \limsup_{i\to\infty} ||m_i - t|| = \varepsilon. \end{aligned} \tag{22}$$

Since $t \in Fix(S)$, therefore using Proposition 1(ii), we get

$$\begin{aligned} ||Sm_i - t|| &\leq ||m_i - t||, \\ \limsup_{i\to\infty} ||Sm_i - t|| &\leq \limsup_{i\to\infty} ||m_i - t|| = \varepsilon. \end{aligned} \tag{23}$$

Again using Lemma 2, we get

$$||m_{i+1} - t|| \leq ||a_i - t||. \tag{24}$$

Using (21) with (22), we get

$$\varepsilon \leq \liminf_{i\to\infty} ||a_i - t||. \tag{25}$$

From (25) and (22), we obtain

$$\lim_{i\to\infty} ||a_i - t|| = \varepsilon. \tag{26}$$

From Lemma 2, we have

$$\begin{aligned} \|a_i - t\| &= \|\xi_i(Sm_i - t) + (1 - \xi_i)(m_i - t)\|, \\ \lim_{i\to\infty} \|a_i - t\| &= \lim_{i\to\infty} \|\xi_i(Sm_i - t) + (1 - \xi_i)(m_i - t)\|. \end{aligned} \tag{27}$$

Using (26) with (27), we get

$$\varepsilon = \lim_{i\to\infty} \|\xi_i(Sm_i - t) + (1-\xi_i)(m_i - t)\|. \tag{28}$$

Using Lemma 1 with (21), (23), and (28), we get

$$\lim_{i\to\infty} ||Sm_i - m_i|| = 0.$$

Now, conversely assume that $\{m_i\}$ is bounded such that $\lim\limits_{i\to\infty} ||Sm_i - m_i|| = 0$. Let $t \in \mathscr{A}(U, \{m_i\})$ and apply Proposition 1(iv). Then, we obtain the following

$$\begin{aligned}\limsup_{i\to\infty} ||m_i - St|| &\leq \left(\frac{3+\alpha}{1-\alpha}\right) \limsup_{i\to\infty} ||m_i - Sm_i|| + \limsup_{i\to\infty} ||m_i - t|| \\ &= \limsup_{i\to\infty} ||m_i - t|| \\ &= r(U, \{m_i\}) \text{ as } t \in \mathcal{A}(U, \{m_i\}).\end{aligned}$$

This implies that $St \in \mathscr{A}(U, \{m_i\})$. As X is a UCBS, so $\mathcal{A}(U, \{m_i\})$ will consist of single element, it further implies that the set $Fix(S)$ is nonempty. □

The following theorem elaborates the weak convergence of our newly proposed iteration process.

Theorem 2 *Assume that S is generalized α–nonexpansive mapping defined on a nonempty closed subset U of a UCBS X, and let $\{m_i\}$ be a sequence generated by (14). If X is equipped with Opial property and $Fix(S)$ is nonempty, then $\{m_i\}$ exhibits weak convergence to a fixed point of S.*

Proof Given that $Fix(S) \neq \emptyset$, so using Theorem 1, we conclude that $\{m_i\}$ is bounded and $\lim\limits_{i\to\infty} ||m_i - Sm_i|| = 0$. It is also given in the theorem statement that X is uniformly convex, therefore X is reflexive. So by Eberlin's theorem one can build a subsequence $\{\varsigma_{i_j}\}$ of sequence of $\{m_i\}$ which weakly converges to $q_1 \in X$. As U is closed and convex, so Mazur's theorem implies that $q_1 \in U$. From Proposition 1(v), we get $q_1 \in Fix(S)$.

Now, we need to prove that $\{m_i\}$ exhibit weak convergence to q_1. Let us suppose that, it is not true, i.e. $\{m_i\}$ fails to converge weakly to q_1. Then, there exists a subsequence $\{\varphi_{i_k}\}$ of $\{m_i\}$ such that $\{\varphi_{i_k}\}$ weakly converges to $q_2 \in U$ and $q_2 \neq q_1$. Again, by using Proposition 1(v), we obtain $q_2 \in Fix(S)$. Now, by using Opial property with Lemma 2, we get

$$\begin{aligned}\lim_{i\to\infty} ||m_i - q_1|| = \lim_{j\to\infty} ||\varsigma_{i_j} - q_1|| &< \lim_{j\to\infty} ||\varsigma_{i_j} - q_2|| \\ &= \lim_{i\to\infty} ||m_i - q_2|| = \lim_{k\to\infty} ||\varphi_{i_k} - q_2|| \\ &< \lim_{k\to\infty} ||\varphi_{i_k} - q_1|| = \lim_{i\to\infty} ||m_i - q_1||.\end{aligned}$$

Which is not possible, therefore $q_1 = q_2$. Hence, $\{m_i\}$ exhibit weak convergence to $q_1 \in Fix(S)$. $\square$

Now, we prove strong convergence theorem for Picard–SP iteration process.

Theorem 3 *Assume that S is generalized α–nonexpansive mapping defined on a nonempty closed and compact subset U of a UCBS X, and let $\{m_i\}$ be a sequence generated by (14). Then, $\{m_i\}$ exhibits strong convergence to a fixed point of S.*

Proof $U \subseteq X$ being a compact and closed and $\{m_i\} \subseteq U$, then there exists a subsequence $\{x_{n_k}\}$ of $\{m_i\}$ such that $\{x_{n_k}\}$ strongly converges to t for some $t \in U$. By applying Proposition 1(iv), we get

$$||x_{n_k} - St|| \leq \left(\frac{3+\alpha}{1-\alpha}\right) ||x_{n_k} - Sx_{n_k}|| + ||x_{n_k} - t||.$$

If we let $k \to \infty$, then $St = t$ which means $t \in Fix(S)$. Since by Lemma 2, $\lim_{i\to\infty} ||m_i - t||$ exists for every $t \in Fix(S)$, so $\{m_i\}$ converges strongly to t. $\square$

Now, we use Condition (I) and prove strong convergence for Picard–SP iteration process.

Theorem 4 *Assume that S is generalized α–nonexpansive mapping defined on a nonempty closed subset U of UCBS X, and $\{m_i\}$ be a sequence generated by (14). If $Fix(S) \neq \emptyset$ and S satisfies Condition (I), then $\{m_i\}$ exhibits strong convergence to a fixed point of S.*

Proof From Lemma 2, we get that $\lim\limits_{i\to\infty} ||m_i - t||$ exists for all $t \in Fix(S)$. Therefore, $\lim\limits_{i\to\infty} d(m_i, Fix(S))$ exists. Suppose that $\lim\limits_{i\to\infty} ||m_i - t|| = \wp$ for some $\wp \geq 0$. For $\wp = 0$, the result is obviously true. Now, if $\wp > 0$, then from the assumption and Condition (I), we have

$$g(d(m_i, Fix(S))) \leq ||m_i - Sm_i||. \tag{29}$$

From Theorem 1, we get

$$\lim_{i\to\infty} ||m_i - Sm_i|| = 0. \tag{30}$$

Since g is non-decreasing function, so by using (30) with (29), we obtain

$$\lim_{i\to\infty} g(d(m_i, Fix(S))) = 0.$$

From the above, we get two subsequences $\{\xi_{i_u}\}$ of $\{m_i\}$ and $\{\eta_u\} \subset Fix(S)$ such that

$$||\xi_{i_u} - \eta_u|| \leq \frac{1}{2^u}, \ \forall u \in \mathbb{N}.$$

Using Lemma 2, we obtain

$$||\xi_{i_{u+1}} - \eta_u|| \leq ||\xi_{i_u} - \eta_u|| \leq \frac{1}{2^u}.$$

Hence,

$$\begin{aligned}||\eta_{u+1}-\eta_u|| &\leq ||\eta_{u+1}-\xi_{i_{u+1}}||+||\xi_{i_{u+1}}-\eta_u||\\ &\leq \frac{1}{2^{u+1}}+\frac{1}{2^u}\rightarrow 0,\ \text{as } u\rightarrow\infty.\end{aligned}$$

Which implies that $\{\eta_u\}$ is Cauchy sequence in $Fix(S)$ and so it converges to t. As $Fix(S)$ is closed, so $t \in Fix(S)$ and then $\{\xi_{i_u}\}$ converges strongly to t. Since by Lemma 2, $\lim\limits_{i\rightarrow\infty}||m_i-t||$ exists, we have $m_i \rightarrow t \in Fix(S)$. □

3 Comparison

Picard–SP iteration process indubitably exhibits a faster convergence rate as compared to other iterations in connection with generalized α–nonexpansive mapping. Observations are explained theoretically and also with the help of a numerical example.

Theorem 5 *Assume that X is any Banach space, and $\emptyset \neq U \subseteq X$ is convex and closed. If a self mapping S defined on U is satisfying (1), and $\{m_i\}$ is a sequence generated by (14), and $\{u_i\}$ is a sequence generated by (9), where $\{\psi_i\}$, $\{\mu_i\}$, $\{\xi_i\}$ are sequences in $(0,1]$ such that $\sum_{i=1}^{\infty}\psi_i=\infty$, then $\{m_i\}$ converges faster than $\{u_i\}$ to a fixed point of S.*

Proof As X is complete and S satisfies (1), so by BCP, S has a unique fixed point in X, say t. Moreover, it is easy to prove that $m_i \rightarrow t$ and $u_i \rightarrow t$ as $i \rightarrow \infty$. Now, by using (14) along with (1), we get

$$\begin{aligned}||a_i-t|| &= ||(1-\xi_i)m_i+\xi_i Sm_i-t||\\ &\leq (1-\xi_i)||m_i-t||+\xi_i||Sm_i-t||\\ &= (1-\xi_i)||m_i-t||+\xi_i||Sm_i-St||\\ &\leq (1-\xi_i)||m_i-t||+\xi_i.\vartheta\|m_i-t\|\\ &= (1-\xi_i(1-\vartheta))||m_i-t||.\end{aligned} \tag{31}$$

Similarly,

$$\begin{aligned}||b_i-t|| &= ||(1-\mu_i)a_i+\mu_i Sa_i-t||\\ &\leq (1-\mu_i)||a_i-t||+\mu_i||Sa_i-t||\\ &\leq (1-\mu_i)||a_i-t||+\mu_i\vartheta||a_i-t||\\ &= (1-\mu_i(1-\vartheta))||a_i-t||.\end{aligned} \tag{32}$$

Using (31) in (32), we get

$$||b_i-t|| \leq (1-\mu_i(1-\vartheta))((1-\xi_i(1-\vartheta))||m_i-t||.$$

Since $0 \leq \vartheta < 1$, therefore $1-\vartheta < 1$ and $\xi_i \in [0,1]$ implies $0 \leq \xi_i(1-\vartheta) < 1$. Hence, $1-\xi_i(1-\vartheta) < 1$, and

$$||b_i - t|| \leq (1-\mu_i(1-\vartheta))||m_i - t||. \tag{33}$$

Using a similar argument, we get

$$||c_i - t|| \leq (1-\psi_i(1-\vartheta))||b_i - t||. \tag{34}$$

Using (33) in (34), we obtain

$$\begin{aligned} ||c_i - t|| &\leq (1-\psi_i(1-\vartheta))((1-\mu_i(1-\vartheta))||m_i - t|| \\ &\leq (1-\psi_i(1-\vartheta))||m_i - t||. \end{aligned} \tag{35}$$

And,

$$||m_{i+1} - t|| = ||Sc_i - t|| \leq \vartheta||c_i - t||. \tag{36}$$

Using (35) in (36), we get

$$\begin{aligned} ||m_{i+1} - t|| &\leq \vartheta(1-\psi_i(1-\vartheta))||m_i - t|| \\ &\leq \vartheta^2(1-\psi_i(1-\vartheta))(1-\psi_{i-1}(1-\vartheta))||m_{i-1} - t||. \end{aligned}$$

Continuing the same way, we get

$$||m_{i+1} - t|| \leq ||m_0 - t||\vartheta^{i+1}\prod_{k=0}^{i}(1-\psi_k(1-\vartheta)). \tag{37}$$

Now, for the sequence $\{u_i\}$ generated by (9), we have the following

$$\begin{aligned} ||w_i - t|| &= ||(1-\xi_i)u_i + \xi_i Su_i - t|| \\ &\leq (1-\xi_i)||u_i - t|| + \xi_i||Su_i - t|| \\ &\leq (1-\xi_i)||u_i - t|| + \xi_i.\vartheta\|u_i - t\| \\ &= (1-\xi_i)||u_i - t|| + \xi_i\vartheta\|u_i - t\| \\ &= ((1-\xi_i(1-\vartheta))||u_i - t||. \end{aligned} \tag{38}$$

Similarly,

$$||v_i - t|| \leq (1-\mu_i(1-\vartheta)||w_i - t||. \tag{39}$$

Using (38) in (39), we get

$$\begin{aligned} ||v_i - t|| &\leq (1-\mu_i(1-\vartheta))((1-\xi_i(1-\vartheta))||u_i - t|| \\ &\leq (1-\mu_i(1-\vartheta))||u_i - t||. \end{aligned}$$

Using a similar argument, we obtain

$$||u_{i+1} - t|| \leq (1 - \psi_i(1 - \vartheta))||v_i - t||. \tag{40}$$

Using (39) in (40), we obtain

$$\begin{aligned} ||u_{i+1} - t|| &\leq (1 - \psi_i(1 - \vartheta))((1 - \mu_i(1 - \vartheta))||u_i - t|| \\ &\leq (1 - \psi_i(1 - \vartheta))||u_i - t|| \\ &\leq (1 - \psi_i(1 - \vartheta))(1 - \psi_{i-1}(1 - \vartheta))||u_{i-1} - t|| \end{aligned} \tag{41}$$

Continuing the same way, we get

$$||u_{i+1} - t|| \leq ||u_0 - t|| \prod_{k=0}^{i}(1 - \psi_k(1 - \vartheta)). \tag{42}$$

Let

$$\mathfrak{a}_i = \vartheta^{i+1} \prod_{k=0}^{i}(1 - \psi_k(1 - \vartheta)),$$

$$\mathfrak{b}_i = \prod_{k=0}^{i}(1 - \psi_k(1 - \vartheta)),$$

and define $\mathfrak{c}_i = \frac{\mathfrak{a}_i}{\mathfrak{b}_i}$.

Now,

$$\begin{aligned} \frac{\mathfrak{c}_{i+1}}{\mathfrak{c}_i} &= \frac{\vartheta^{i+2} \prod_{k=0}^{i+1}(1 - \psi_k(1 - \vartheta))}{\prod_{k=0}^{i+1}(1 - \psi_k(1 - \vartheta))} \cdot \frac{\prod_{k=0}^{i}(1 - \psi_k(1 - \vartheta))}{\vartheta^{i+1} \prod_{k=0}^{i}(1 - \psi_k(1 - \vartheta))} \\ &= \vartheta < 1 \quad \because \vartheta \in [0, 1). \end{aligned}$$

So, by Ratio test (i.e., suppose for any series $\sum_{i=1}^{\infty} x_i$, if $\lim\limits_{i\to\infty} \frac{x_{i+1}}{x_i} < 1$, then $\sum_{i=1}^{\infty} x_i$ exists), we can conclude that $\sum_{i=1}^{\infty} \mathfrak{c}_i$ exists. Moreover,

$$\lim_{i\to\infty} \mathfrak{c}_i = 0 \Rightarrow \lim_{i\to\infty} \frac{\mathfrak{a}_i}{\mathfrak{b}_i} = 0. \tag{43}$$

Using (43) with Definition 11, we can conclude that $\{m_i\}$ converges faster than $\{u_i\}$ to the fixed point t of S. □

In order to demonstrate the improved performance of the proposed Picard–SP iteration process, we consider a numerical example in which we compare our method with the Noor (8), SP (9), P (10), D (11), Picard–Noor (12) and Picard–S^* (13) and iteration processes.

Example 1 Let $X = \mathbb{R}$ be a Banach space and $U = [0, 7]$ is a subset of X equipped with the norm $||p|| = |p|$. Define a function $S : U \to U$ as

$$Sp = \begin{cases} \frac{p+24}{5}, & \text{if } p \in [0, 6], \\ 5, & \text{if } p \in (6, 7]. \end{cases} \tag{44}$$

S is generalized α–nonexpansive but does not satisfy Condition (C).

Firstly, we show that S does not satisfy Condition (C). For $p = \frac{29}{5}$ and $\mathfrak{z} = \frac{31}{5}$, we have

$$Sp = \frac{149}{25},$$
$$S\mathfrak{z} = 5,$$
$$\|p - \mathfrak{z}\| = \left|\frac{29}{5} - \frac{31}{5}\right| = \frac{2}{5}.$$

Now,

$$\frac{1}{2}\|p - Sp\| = \frac{1}{2}\left|\frac{29}{5} - \frac{149}{25}\right| = \frac{2}{25} < \frac{2}{5} = \|p - \mathfrak{z}\|.$$

Then,

$$\|Sp - S\mathfrak{z}\| = \left|\frac{149}{25} - 5\right| = \frac{24}{25} > \frac{2}{5} = \|p - \mathfrak{z}\|.$$

Therefore, S fails to satisfy Condition (C).

Next, we show that S is generalized α–nonexapnsive mapping. Choose $\alpha = \frac{1}{2}$. Let us consider cases:

1. $p, \mathfrak{z} \in [0, 6]$. Then,

$$\|Sp - S\mathfrak{z}\| = \left|\frac{p + 24}{5} - \frac{\mathfrak{z} + 24}{5}\right| = \frac{1}{5}\|p - \mathfrak{z}\|.$$

Now,

$$\begin{aligned} \alpha\|p - S\mathfrak{z}\| + \alpha\|\mathfrak{z} - Sp\| + (1 - 2\alpha)\|p - \mathfrak{z}\| &= \frac{1}{2}\left|p - \frac{\mathfrak{z} + 24}{5}\right| + \frac{1}{2}\left|\mathfrak{z} - \frac{p + 24}{5}\right| \\ &\geq \frac{1}{2}\left|\frac{5p - \mathfrak{z} - 24}{5} + \frac{5\mathfrak{z} - p - 24}{5}\right| = \frac{1}{2}\left|\frac{4p + 4\mathfrak{z} - 48}{5}\right| \\ &\geq \frac{1}{5}\|p - \mathfrak{z}\| = \|Sp - S\mathfrak{z}\|. \end{aligned}$$

Thus, (5) is satisfied.

2. $p, \mathfrak{z} \in (6, 7]$. Then,

$$\|Sp - S\mathfrak{z}\| = |1 - 1| = 0.$$

Now,

$$\alpha\|p - S\mathfrak{z}\| + \alpha\|\mathfrak{z} - Sp\| + (1-2\alpha)\|p - \mathfrak{z}\| = \frac{1}{5}|p-5| + \frac{1}{5}|\mathfrak{z}-5| + \left(1 - 2\cdot\frac{1}{5}\right)\|p-\mathfrak{z}\| \geq \|Sp - S\mathfrak{z}\|.$$

Thus, (5) is satisfied.

3. $p \in [0, 6]$ and $\mathfrak{z} \in (6, 7]$. Then,

$$\|Sp - S\mathfrak{z}\| = \left|\frac{p+24}{5} - 5\right| = \frac{1}{5}|p-1|.$$

Now,

$$\begin{aligned}\alpha\|p - S\mathfrak{z}\| + \alpha\|\mathfrak{z} - Sp\| + (1-2\alpha)\|p - \mathfrak{z}\| &= \frac{1}{2}|p-5| + \frac{1}{2}\left|\mathfrak{z} - \frac{p+24}{5}\right| \\ + \left(1 - 2\cdot\frac{1}{2}\right)\|p-\mathfrak{z}\| &= \frac{1}{2}|p-5| + \frac{1}{2}\left|\frac{5\mathfrak{z} - p - 24}{5}\right| \\ \geq \frac{1}{2}\left|\frac{4p + 5\mathfrak{z} - 49}{5}\right| &\geq \frac{1}{5}|p-1| = \|Sp - S\mathfrak{z}\|.\end{aligned}$$

Thus, (5) is satisfied.

Hence, we conclude that S given in (44) is generalized $\frac{1}{2}$–nonexpansive mapping.

In the numerical example, we set $\psi_i = 0.75$, $\mu_i = 0.85$, $\xi_i = 0.80$, and the initial value $m_0 = 2$ for all the considered iteration schemes, i.e., the Picard–SP, Noor, SP, P, D, Picard–S*, and Picard–Noor iterations. We set the stopping criterion $||m_i - m_{i+1}|| < 10^{-8}$. The obtained results are presented in Table 1 and Figs. 1 and 2.

From the obtained results, we see that after the first iteration, the value calculated using the Picard–SP (5.96313600) is closer to the fixed point (i.e., 6) as compared to the first iteration of the other iterative processes. The closest fixed point approximation among the methods from the literature can be observed for the Picard–S* iteration. In the subsequent iterations, we see that each iteration scheme gets closer to the fixed point but with various speeds. The fastest method is the proposed Picard–SP iteration, which found the fixed point in 4 iterations. The second best method is the Picard–S* iteration, which needed 5 iterations. For the D and SP iterations, we observe a very similar behavior. We can also notice a similar behavior for the Picard–Noor and P iteration processes (their plots are very close to each other). The worst convergence speed is observed for the Noor iteration, which needed 15 iterations to find the fixed point.

Table 1 Iterates m_i generated by various iteration processes for the mapping S considered in Example 1, and the starting point $m_0 = 2$

i	Picard–SP iteration	Picard–S* iteration	SP iteration	D iteration	Picard–Noor iteration	P iteration	Noor iteration
0	2	2	2	2	2	2	2
1	5.96313600	5.94905600	5.81568000	5.87328000	5.77465600	5.73088000	4.87328000
2	5.99966030	5.99935120	5.99150650	5.99598551	5.98730500	5.98189361	5.68262551
3	5.99999686	5.99999170	5.99960860	5.99987282	5.99928480	5.99878180	5.91060195
4	6	5.99999989	5.99998200	5.99999597	5.99995970	5.99991804	5.97481836
5	6	6	5.99999916	5.99999987	5.99999770	5.99999448	5.99290684
6	6	6	6	6	5.99999987	5.99999962	5.99800200
7	6	6	6	6	6	6	5.99943720
8	6	6	6	6	6	6	5.99984147
9	6	6	6	6	6	6	5.99995534
10	6	6	6	6	6	6	5.99998742
11	6	6	6	6	6	6	5.99999646
12	6	6	6	6	6	6	5.99999900
13	6	6	6	6	6	6	5.99999972
14	6	6	6	6	6	6	5.99999992
15	6	6	6	6	6	6	6

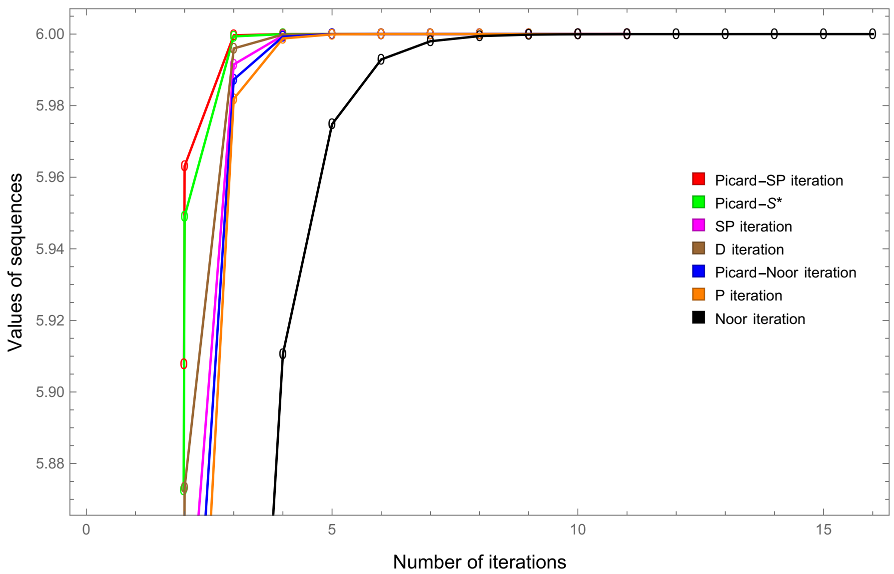


Fig. 1 Convergence behavior of Picard–SP (14), Noor (8), SP (9), P (10), D (11), Picard–S* (13) and Picard–Noor (12) iteration processes corresponding to Table 1

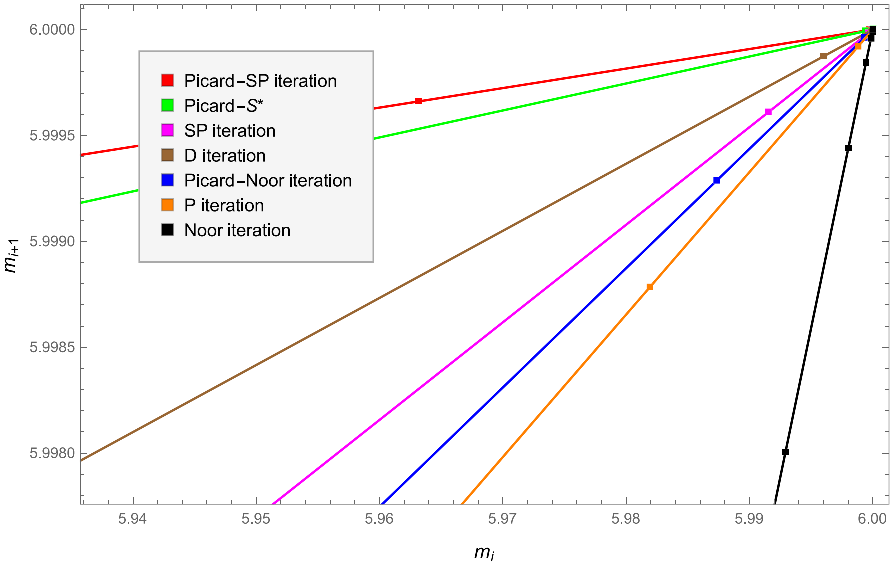


Fig. 2 Convergence analysis of Picard–SP (14), Noor (8), SP (9), P (10), D (11), Picard–S* (13) and Picard–Noor (12) iteration processes with respect to the values given in Table 1

4 Comparison via polynomiography

In this section, we present an empirical comparison of the Picard–SP iteration process with the Noor, SP, P, D, Picard–S* and Picard–Noor iteration processes for fixed points approximation of Newton's iteration operator via the so-called polynomiography. The term polynomiography was introduced by Kalantari in [17], and it is defined as "the art and science of visualization in approximation of the zeros of complex polynomials, via fractal and non-fractal images created using the mathematical convergence properties of iteration functions". An image generated with polynomiography is called a polynomiograph. The methods of polynomiography are widely used in the comparison and analysis of various kinds of iteration processes, see for example [14, 20, 25, 35, 36]. Polynomiographs can also produce artistic patterns [11].

The famous Newton's method [18] for a complex polynomial Q is defined as

$$p_{n+1} = p_n - \frac{Q(p_n)}{Q'(p_n)}, \tag{45}$$

where $p_0 \in \mathbb{C}$ is a starting point. Newton's iteration process can be expressed in the form of a fixed point iterative process as follows:

$$p_{n+1} = S(p_n), \tag{46}$$

where $S(p) = p - Q(p)/Q'(p)$. We see that (46) is the Picard iteration. If the iteration process converges to a fixed point $\mathfrak{z} \in \mathbb{C}$ of S, then

$$\mathfrak{z} = S(\mathfrak{z}) = \mathfrak{z} - \frac{Q(\mathfrak{z})}{Q'(\mathfrak{z})}. \tag{47}$$

Thus, $\mathfrak{z}$ is a root of Q because $Q(\mathfrak{z})/Q'(\mathfrak{z}) = 0 \iff Q(\mathfrak{z}) = 0$.

Now, instead of the Picard iteration, we can use other iteration processes, e.g., the introduced Picard–SP iteration or other iteration processes defined in Sec. 1.

To generate polynomiographs, we use the algorithm presented as a pseudocode in Algorithm 1. In the algorithm, we use the so-called iteration coloring to color the points [18]. In this type of coloring, for each starting point, we assign color according to the number of performed iterations, which can be interpreted as the speed of convergence. Therefore, this type of polynomiograph presents the speed of convergence of the root-finding method. Moreover, using the polynomiograph generated using Algorithm 1, we can calculate an average number of iterations (ANI) [11].

Algorithm 1 Generation of a polynomiograpgh.

Input: $Q \in \mathbb{C}[Z]$, $\deg Q \geq 2$ – polynomial; I – iteration process; $A \subset \mathbb{C}$ – area; N – the maximum number of iterations; ε – accuracy; $colors$ – color map.

Output: Polynomiograph for the complex-valued polynomial Q within the area A.

```
for z_0 ∈ A do
    n = 0
    while |Q(z_n)| > ε and n < N do
        z_{n+1} = I(z_n, Q)
        n = n + 1
    Map n to a color from the color map colors and color z_0
```

In the considered example, we generate polynomiographs for a cubic polynomial $Q(z) = z^3 - 1$ and various iteration processes, namely the introduced Picard–SP iteration and the Noor, SP, P, D, Picard–Noor, and Picard–S* iterations known in the literature. The polynomiographs were generated for three different settings of values of the iterations' parameters: (1) $\xi_i = 0.01$, $\mu_i = 0.01$, $\psi_i = 0.01$, (2) $\xi_i = 0.5$, $\mu_i = 0.5$, $\psi_i = 0.5$, (3) $\xi_i = 0.8$, $\mu_i = 0.8$, $\psi_i = 0.8$. All the other parameters needed to generate the polynomiographs were common: $A = [-2, 2]^2$, $N = 15$, $\varepsilon = 0.001$ and the color map presented in Fig. 3.

The generated polynomiographs for the three settings of the parameters are presented in Figs. 4, 5, and 6, whereas the ANI values calculated from the polynomiographs are gathered in Table 2. For low values of the parameters (Fig. 4), we

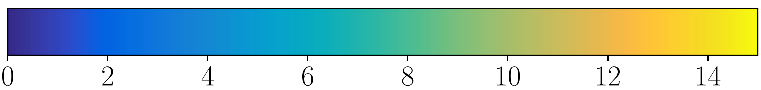
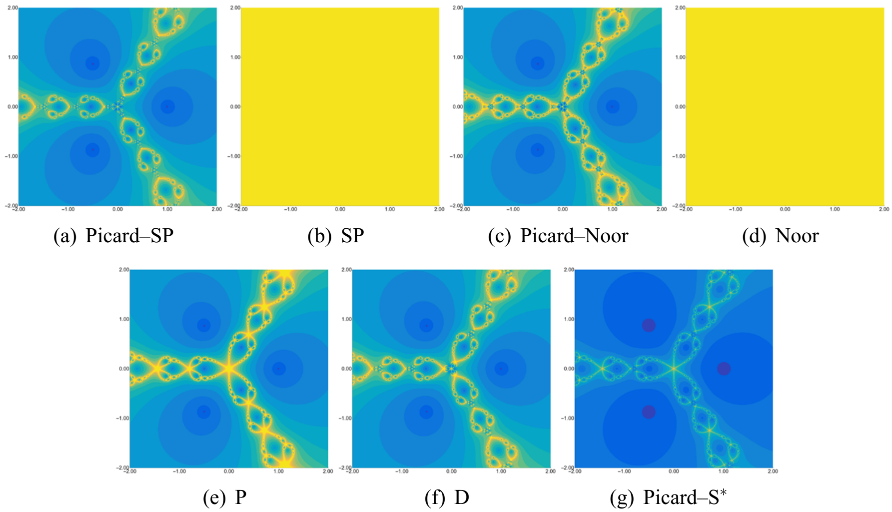


Fig. 3 Color map used in the examples

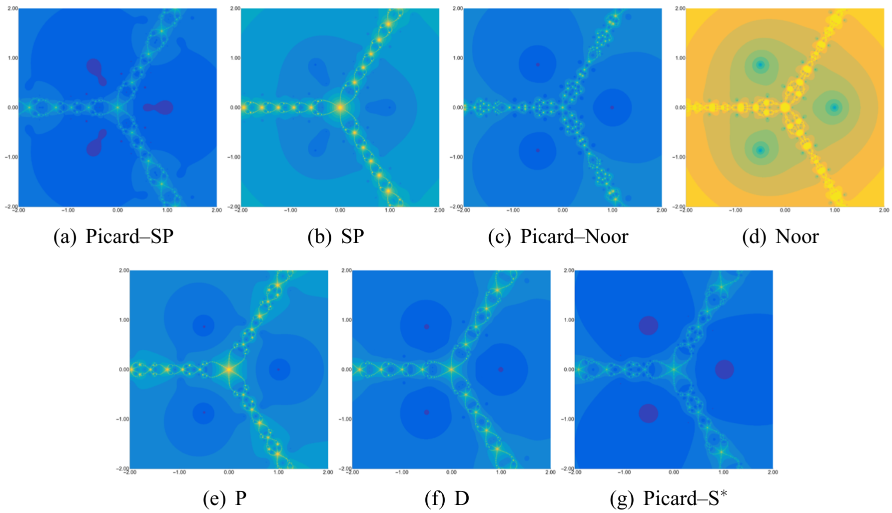


(a) Picard–SP (b) SP (c) Picard–Noor (d) Noor

(e) P (f) D (g) Picard–S*

Fig. 4 Polynomiographs generated by various iteration processes with the parameters $\xi_i = 0.01$, $\mu_i = 0.01$, $\psi_i = 0.01$

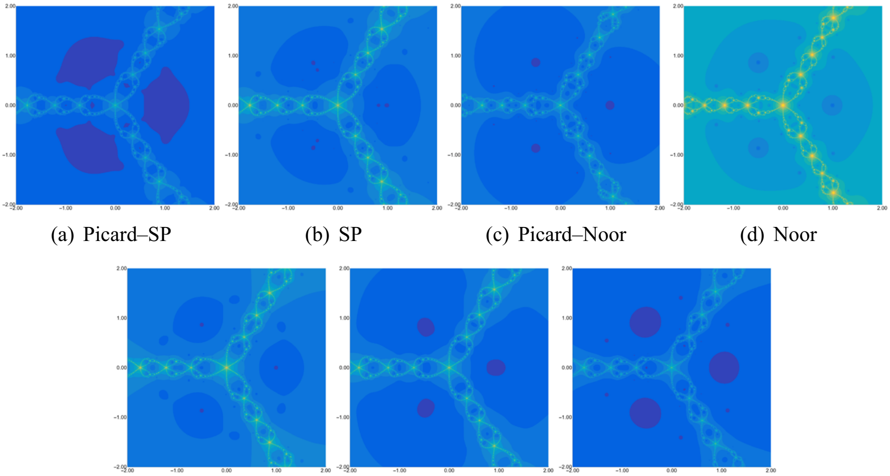
see that two of the iterations have not converged to any of the three roots of Q, i.e., we see a uniform yellow color, which corresponds to the maximal of 15 iterations. For the other iterations, we see a different speed of convergence. Based on the visual analysis, we can observe that the fastest speed of convergence is obtained by the Picard–S*, followed by the proposed Picard–SP iteration and the D and P iterations. These observations are confirmed by the ANI values in Table 2. The lowest ANI value 1.165 is



(a) Picard–SP (b) SP (c) Picard–Noor (d) Noor

(e) P (f) D (g) Picard–S*

Fig. 5 Polynomiographs generated by various iteration processes with the parameters $\xi_i = 0.5$, $\mu_i = 0.5$, $\psi_i = 0.5$



(a) Picard–SP (b) SP (c) Picard–Noor (d) Noor

(e) P (f) D (g) Picard–S*

Fig. 6 Polynomiographs generated by various iteration processes with the parameters $\xi_i = 0.8$, $\mu_i = 0.8$, $\psi_i = 0.8$

obtained by the Picard–S* iteration, followed by the Picard–SP (5.502), D (5.659) and P (5.966) iterations. Moreover, we can observe that the addition of the Picard step in the SP and Noor iterations significantly improves the speed of convergence.

For polynomiographs for the second parameters setting presented in Fig. 5, we see that the slowest speed of convergence is obtained by the Noor iteration. The polynomiograph contains yellowish colors, indicating a high number of performed iterations. When we look at the polynomiograph for the SP iteration, we see a much faster speed of convergence in comparison to the Noor iteration. The fastest among the analyzed iterations is the Picard–SP iteration. In the polynomiographs, we can observe more darker blue colors than in the case of the other iteration processes, which shows a smaller number of performed iterations. The ANI values confirm this observation because the lowest value equal to 2.503 is obtained by the Picard–SP iteration. The second best iteration, in terms of speed of convergence, is the Picard–S* iteration

Table 2 ANI values calculated from polynomiographs presented in Figs. 4–6

Iteration	$\xi_i = \mu_i = \psi_i = 0.01$	$\xi_i = \mu_i = \psi_i = 0.5$	$\xi_i = \mu_i = \psi_i = 0.8$
Picard–SP	5.502	2.503	2.053
SP	15	4.967	3.027
Picard–Noor	5.795	3.378	2.525
Noor	15	12.211	5.791
P	5.966	4.066	3.268
D	5.659	3.280	2.631
Picard–S*	1.165	2.520	2.255

(2.520), and the third best is the D iteration (3.280). The highest value of ANI equal to 12.211 is obtained by the Noor iteration. As for the first parameters' setting, we can observe that the addition of the Picard step to the SP and Noor iterations improved their speed of convergence.

In the last parameters setting, we use high values of the parameters. Like for the other two parameter settings, for the polynomiographs in Fig. 6, we see that the slowest speed of convergence is obtained by the Noor iteration. On the other side, the fastest speed of convergence is again obtained by the Picard–SP iteration. In the case of each of the polynomiographs, we can observe that the colors are darker than for the two other parameter settings. This shows that for higher values of the parameters, all the iterations need fewer iterations to find the roots. We can also observe this by looking at the values of ANI in Table 2. We see that the lowest ANI value equal to 2.053 is obtained by the Picard–SP iteration for high values of the parameters. The lowest values of ANI for the other iterations are also obtained for high values of the parameters. The only exception is the Picard–S* iteration, for which the best result is obtained for the lowest values of the parameters.

5 Conclusions

In this paper, we have investigated the convergence analysis of the newly proposed Picard–SP iteration process and its efficient utilization for fixed point estimation of generalized α-nonexpansive mappings. Efficacy is illustrated numerically as well as theoretically. The results show the superiority of the proposed Picard–SP iteration over the Noor, SP, P, D, Picard–S* and Picard–Noor iteration processes. Moreover, we empirically compared the Picard–SP iteration with the Noor, SP, P, D, Picard–S* and Picard–Noor iteration processes in the root-finding problem via Newton's method. For the comparison, we used polynomiography. Again, the results show a better speed of convergence of the Picard–SP iteration.

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Declarations

Conflicts of Interest The authors declare that they have no conflict of interest.

Ethical Approval Not Applicable.

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